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On Stability Problems of Supply Networks Constrained With Transport Delay[1]

Transportation is one of the most crucial components in supply networks. In transportation lines, there exists a finite time between products leaving a point and arriving to another point in the supply network. This period of time is the delay, which accompanies all transportation lines present in the entire network. Delay is a well-known limitation, which is inevitable and pervasive in the network causing synchronization problems, fluctuating or excessive inventories, and lack of robustness of inventories against cyclic perturbations. The end results of such undesirable effects directly reflect to costs. This paper is motivated to reveal the mechanisms leading to these problems by analytically characterizing qualitative behavior of supply network dynamics modeled by continuous-time differential equations. The presence of delay forms the main challenge in the analysis and this is tackled by developing/utilizing the tools emerging from delay systems and control theory. While the backbone of the paper addresses the qualitative behavior in presence of a single delay representing delays in all transportation paths, it also reveals how to choose production rates and transportation delay without inducing any undesirable effects mentioned. Thorough cases studies with single and multiple delays are presented to demonstrate the effectiveness of the approaches proposed.
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1 Introduction

Supply networks [1–6] are an ensemble of interconnections of customers, suppliers, manufacturing units, companies, and sources (Fig. 1). While supplies flow along the directed links of these networks to satisfy the changing demand of customers (solid lines in Fig. 1), the information concerning the product orders flows in the opposite direction (dashed lines in Fig. 1). One of the main objectives in supply network management is to control individual production rates such that inventories maintain their desired levels when responding to customer demands. Although this seems to be a simplistic proposition, supply network management is challenging [6–10].

Supply networks inherently carry major limitations in their responsiveness and synchronization [6,11–13] since their dynamics respond to demand variations after a period of time needed for information collection, perception, decision making, communication, and transportation. This period of time, which is known as delay, hampers supply network management [3,4,6,11–14]. Since transportation is one of the *major* components of supply networks, the delay arising due to transport of products is the focus in this paper with particular interest in analyzing how a delay affects inventory dynamics.

The most undesirable effect delay brings to supply networks is that inventories may become oscillatory, underdamped, and even unstable as a result of cross coupling between delayed information and decision making (feedback loop). Although these effects seem to be similar to those detected in Ref. [15], the qualitative and quantitative conditions that give rise to these similarities with and without delays are quite different.

This paper is aligned with the philosophy of "system's thinking" adapted earlier in Refs. [3,4,6,9,16,57]. We depart from a *delay-free* mathematical model [15] of a supply network dynamics, incorporate a transportation delay, and connect the arising model with analytical tools we develop via the advancements in the field of *time delay systems* and *control*. Analytical tools are derived in order to explain qualitative behavior of supply networks and inventory variations in the presence of transportation delay. For this umbrella objective, we first take all transportation delays in the network identical, σ. Particularly we wish to reveal the so-called stability maps on which stability versus instability of the supply network is characterized with respect to the intrinsic network parameters and this delay σ. In addition to these, we put light on *bullwhip effects* [6,17–20,56] and we explain how supply network becomes more *prone* to exhibiting bullwhip in the presence of delay σ.

What differentiates this work from Refs. [3,4,9,13] is that we develop *analytical approaches* in analyzing *higher* degrees of freedom inventory dynamics in which transportation *delay* σ is taken into account as a first-in-first-out (FIFO) parameter. It is known, however, that analyzing stability with *multiple* delay presence is orders of magnitude more difficult due to the *NP* hard complexity of the problem [21]. A brief discussion on recently developed methodologies in treating stability problems with multiple delays is included and a motivational problem with *four independent* transportation delays is treated by advanced clustering with frequency sweeping (ACFS) methodology [22] developed recently. Arising stability results, simulations, and interpretations are provided.

To our best knowledge, the final results of our study in the context of supply networks are new: (a) We analyze the effects of transportation delay to stability and bullwhip; (b) we analytically characterize stability (Sec. 3) and bullwhip phenomenon (Sec. 4) with respect to the eigenvalues of a characteristic matrix, parameters defining the network and delay; (c) with the tools developed, we explain analytically how the network becomes more prone to exhibit bullwhip effects in the presence of delays; (d) we present

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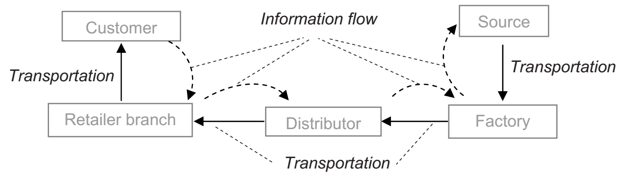

Fig. 1 A generic supply-demand flow from source to customers, inspired by Sterman [6]

some preliminary results where multiple delays are considered. Case studies (Sec. 5) are provided to demonstrate the effectiveness and impacts of obtained results.

Notations. $\mathbb{R}$ ($\mathbb{C}$ and $\mathbb{Z}$) denotes the real (complex and integer) numbers. $\mathbb{R}_-$ and $\mathbb{R}_+$ ($\mathbb{Z}_-$ and $\mathbb{Z}_+$) is the set of negative and positive real (integer) numbers, respectively; $\mathbb{C}_+$ ($\mathbb{C}_-$) corresponds to the right (left) half complex plane. The imaginary axis of $\mathbb{C}$ is denoted by $j\mathbb{R}$, where $j=\sqrt{-1}$. We use $s\in\mathbb{C}$ for the Laplace variable, whose real and imaginary parts are $\Re(s)$ and $\Im(s)$, respectively. Boldface is used to indicate vectors and matrices. The entry in the ith row and the jth column of a matrix $\mathbf{M}\in\mathbb{R}^{n\times n}$ is represented by M_{ij}, the identity matrix with appropriate dimensions is $\mathbf{I}$, and a vector $\mathbf{v}$ with entries v_i is denoted by $\mathbf{v}=(v_i)$.

2 Preliminaries

In this section, we first present the problem formulation and a brief mathematical description of the *delay-free* supply network model borrowed from Ref. [15]. We take a supply network of n suppliers i delivering products to other suppliers μ or to costumers. It is assumed in this study that each supplier delivers one type of product only. More complicated network structures are left for future work. The rate at which supplier i delivers products to and consumes product from supplier μ is given by $d_{\mu i}X_i(t)$ and $c_{i\mu}X_i(t)$, respectively, where $X_i(t)>0$ denotes the production rate. The coefficients $c_{i\mu}$ define an input matrix $\mathbf{C}$ and the entries $d_{i\mu}$ define an output matrix $\mathbf{D}$, where $0\le d_{i\mu},c_{i\mu}\le 1$. Since at most 100% of the products produced by all μ suppliers can be delivered to i suppliers, the physical constraint $\Sigma_\mu c_{i\mu}\le 1$ holds. The inventory level $N_i(t)$ at supplier i changes as

$$\frac{dN_i(t)}{dt}=\sum_{\mu}(d_{i\mu}-c_{i\mu})X_\mu(t)+Y_i(t) \tag{1}$$

where the external demand is denoted by $Y_i(t)$. In order to keep the inventory at some desired level $\bar{N}_i$, any changes in the demand $Y_i(t)$ require an adaptation of the production rates $X_i(t)$ governed by

$$\frac{dX_i(t)}{dt}=\frac{1}{T}(W_i(N_i(t),dN_i(t)/dt)-X_i(t)) \tag{2}$$

where T is the time constant, W_i is a nonlinear function dependent on the inventories $N_i(t)$ and their change in time $dN_i(t)/dt$. The function W_i is assumed to be non-negative and it decreases with increasing inventories.

It is often implicitly needed in queuing theory to reveal how the stationary state (equilibrium) of the supply network behaves (in a stable or unstable regime). The state of the equilibrium is also of practical interest as one needs to reveal how a perfectly equilibrated inventory dynamically behaves under small perturbations and in the presence of delays, see Refs. [3,4,9] and references therein.

After normalizing time t by T, $\tau=t/T$ and linearization of Eqs. (1) and (2), we have

$$\frac{d^2\mathbf{x}(\tau)}{d\tau^2}+[\mathbf{I}+B(\mathbf{D}-\mathbf{C})]\frac{d\mathbf{x}(\tau)}{d\tau}+A(\mathbf{D}-\mathbf{C})\mathbf{x}(\tau)=A\mathbf{y}(\tau)+B\frac{d\mathbf{y}(\tau)}{d\tau} \tag{3}$$

where $\mathbf{x}(\tau)=(x_i(\tau))$ is the vector carrying *variations* in inventories and production rates and $\mathbf{y}(\tau)=(y_i(\tau))$ is the vector with entries being variations in consumption rates; A and B are both positive constants and they represent the local derivatives of the nonlinear function W_i with respect to N_i and dN_i/dt, respectively, see Ref. [15] for details.

Since each supplier is assumed to produce only one type of product i, the output matrix $\mathbf{D}$ has a special structure—it is diagonal. Furthermore, we shall investigate the case when output is at its *maximum delivery rate* setting $d_{i\mu}=1$ or $\mathbf{D}=\mathbf{I}$. Other cases when $d_{i\mu}$ takes different numerical values can be easily adapted to the framework developed in this paper. We wish state that the choice[2] of a diagonal matrix $\mathbf{D}$ also allows a Jordan decomposition, which in return enables reconstructing the supply network as a connection of quasisupply dynamics arranged in a *chain* [15]. Furthermore, input matrix $\mathbf{C}$ can be transformed in either a diagonal or a Jordan normal form $\mathbf{J}$ by a matrix $\mathbf{T}$, $\mathbf{T}^{-1}\mathbf{C}\mathbf{T}=\mathbf{J}$, and due to the particular form of $\mathbf{C}$ its eigenvalues or equivalently the entries J_{ii} have the property $0\le|J_{ii}|\le 1$. Defining $\boldsymbol{\mu}(\tau)=(\mu_i(\tau))$ $=\mathbf{T}^{-1}\mathbf{x}(\tau)$ and $\mathbf{h}(\tau)=(h_i(\tau))=\mathbf{T}^{-1}(A\mathbf{y}(\tau)+Bd\mathbf{y}(\tau)/d\tau)$, we obtain a set of coupled second-order differential equations:

$$\frac{d^2\mu_i(\tau)}{d\tau^2}+2\gamma_i\frac{d\mu_i(\tau)}{d\tau}+\omega_i^2\mu_i(\tau)=b_i\left(A\mu_{i+1}(\tau)+B\frac{d\mu_{i+1}(\tau)}{d\tau}\right)+h_i(\tau) \tag{4}$$

$$\gamma_i=\frac{1+B(1-J_{ii})}{2},\quad \omega_i=\sqrt{A(1-J_{ii})},\quad b_i=J_{i,i+1} \tag{5}$$

which can be seen as a chain of connected damped oscillators with damping constants γ_i, natural frequencies ω_i, and external forcing $h_i(\tau)$. The remaining terms on the right hand side of Eq. (4) appear due to the mutual interactions of the suppliers with each other, and as per Jordan decomposition, b_i is either 1 or 0.

3 Stability Analysis With Transport Delays

In order to analyze the stability of the supply network, one needs to understand how the delay enters in the differential equation (4) and how delay is mathematically described. After incorporating delay σ into Eq. (4), analytical approaches that are free of conservatism are developed to assess the stability of the supply network with respect to σ. Arising managerial strategies are discussed along with sensitivity of the network to bullwhip effects. This section concludes with discussions in analyzing supply networks with multiple delays σ_v, $v=1,\ldots,\bar{v}$, where $\bar{v}$ is the number of transportation lines.

3.1 Delays. In this study, we consider delays as *pure* time translations, at an amount of σ, where products shipped at time t arrive their destinations at time $t+\sigma$ (Fig. 2). This choice is in compliance with the earlier work [3,4,9] and references therein, where appropriateness of utilizing pure delays is pointed out. Depending on the physics of the problem, one may also choose distribution functions, such as gamma and Erlang functions as discussed in Refs. [6,23]. The main difference between pure delays and distributed delays is as follows [2,3,6,24,25]. Physically, pure delays correspond to FIFO type of behavior, while distributed delays model some degree of mixing where first-in is not necessarily first-out. Mathematically speaking, a pure delay leads to infinite-dimensional systems described by functional differential equations and distributed delays lead to integrodifferential equations, which can correspond to finite- or infinite-dimensional sys-

[2]In Sec. 3.4, we shall show other directions to relax these assumptions.

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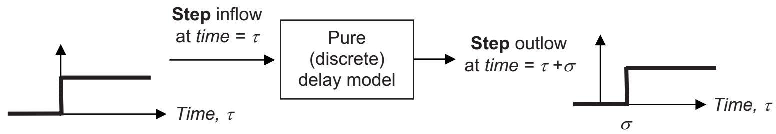

Fig. 2 Pure (discrete) delay modeling and its effects between an input and an output

tems depending on the distribution law [26,27]. Often, however, distributed delays are only approximations of pure delays, and in the limiting case when variance of a distribution becomes zero, the distribution becomes an infinite-dimensional function precisely corresponding to pure delay [6,23]. For other delay models of time-dependent and state-dependent type, please refer to Refs. [6,28–30].

3.2 Governing Dynamics With Pure Delays. As discussed earlier, we wish to study delay effects in product deliveries, hence pure (discrete) delay affects the terms in the output matrix **D**. Reconstructing Eq. (3), the governing dynamics of the supply network with delay σ is obtained as

$$\frac{d^2\mu_i(\tau)}{d\tau^2} + B\frac{d\mu_i(\tau-\sigma)}{d\tau} + (1-BJ_{ii})\frac{d\mu_i(\tau)}{d\tau} + A\mu_i(\tau-\sigma) - AJ_{ii}\mu_i(\tau) = b_i\left(A\mu_{i+1}(\tau) + B\frac{d\mu_{i+1}(\tau)}{d\tau}\right) + h_i(\tau) \quad (6)$$

As expected, one recovers Eq. (4) by setting $\sigma=0$ in the above delay differential equation (DDE). See Ref. [5] for some of the preliminary steps along these lines.

In the following, we will perform *stability analysis* and investigate the sensitivity of *bullwhip effects* in the DDE. These will result how input rates J_{ii} and parameters defining the local variations of the inventories, A and B, affect supply network with respect to delay σ.

3.3 Stability Analysis. Stability analysis is adapted by following a frequency sweeping idea [31–34]. This idea enables a nonconservative approach in obtaining stability features of the supply network with respect to delay σ. In order to assess the stability of the linear supply network dynamics (6), we first obtain its characteristic equation $F(s,\sigma)=0$ in Laplace domain using its homogeneous part and assuming zero initial conditions:

$$F(s,\sigma) = \prod_{i=1}^{n}[f_i(s,\sigma)] = 0$$
$$f_i(s,\sigma) = (s^2 + (1-BJ_{ii})s - AJ_{ii}) + (Bs+A)e^{-\sigma s} \quad (7)$$

where n is the number of quasisupply nodes. The stability analysis problem is reduced to developing a technique to verify the stability condition for $f_i(s,\sigma)=0$, $i=1,\ldots,n$. This verification is not straightforward since the characteristic equation is transcendental and thus it has an infinite number of roots with particular properties. These complications open new research directions [35] in mathematics [21,27,28], numerics [36], and engineering [32,37–39] that attempt to assess the stability of similar infinite-dimensional problems with respect to system parameters and delays. Below, we develop tools to tackle the infinite-dimensional complexity of $F(s,\sigma)=0$ toward assessing the stability of the supply network.

The continuity principle [40,41] states that the rightmost root of $F(s,\sigma)=0$ moves continuously in $\mathbb{C}$ with respect to continuous variations in σ. Hence, stability or instability of the dynamics *may* change only when a characteristic root crosses the imaginary axis at $s=\lambda j$ ($\lambda \geq 0$ without loss of generality) for some σ^*. If one may propose a technique to detect all ($s=\lambda^* j$, σ^*) pairs for which $f_i(\lambda^* j,\sigma^*)=0$, $i=1,\ldots,n$, then it is possible to calculate toward which direction these characteristic roots move across the imaginary axis (either toward $\mathbb{C}_-$ or $\mathbb{C}_+$) by using the following sensitivity expression:

$$S_i(s,\sigma) = \frac{ds}{d\sigma} = -\frac{\partial f_i(s,\sigma)}{\partial\sigma}\left(\frac{\partial f_i(s,\sigma)}{\partial s}\right)^{-1}$$

If $\Re(S_i(\lambda^* j,\sigma^*)) > 0$ (<0), then $s=\lambda^* j$ root on the imaginary axis will move toward right half complex plane favoring instability (or left half complex plane favoring stability) as σ^* increases infinitesimally. With this information, one can easily account how many unstable roots exist in $\mathbb{C}_+$ for any given delay σ. Obviously, no roots in $\mathbb{C}_+$ indicate stability, otherwise instability.

3.3.1 Detection of Imaginary Roots. In order to detect the imaginary roots of $f_i(s,\sigma)=0$, a practical technique, which is based on a geometrical characterization, is deployed. A similar approach was followed in Refs. [42,43]. We first solve for the exponential term and substitute $s=\lambda j$,

$$e^{\sigma\lambda j} = \frac{B\lambda j + A}{\lambda^2 + (BJ_{ii}-1)\lambda j + AJ_{ii}} \quad (8)$$

which can be analyzed for its magnitude and argument in complex domain $\mathbb{C}$.

Notice that $|(e^{\sigma s})_{s=\lambda j}|=1$ for $\forall \sigma \in \mathbb{R}$ depicts a unit circle on $\mathbb{C}$. Therefore, for an imaginary root $s=\lambda j$ of $f_i(s,\sigma)$ to exist, the unit circle and the curve created by the magnitude condition of the right hand side of Eq. (8) should intersect. Denote this curve by $\chi(\lambda)$,

$$\chi(\lambda) = \sqrt{\frac{B^2\lambda^2 + A^2}{(\lambda^2 + AJ_{ii})^2 + (BJ_{ii}-1)^2\lambda^2}} \quad (9)$$

For given parameters A, B, and J_{ii}, the geometry of the curve $\chi(\lambda)$ can be checked if it intersects with the unit circle as λ sweeps from 0 to $+\infty$. The intersections take place only when $\chi(\lambda^*)=1$. The way of expressing the problem as a geometric interaction between a unit circle and a frequency dependent curve $\chi(\lambda)$ enables a convenient way of solving for λ^* by decoupling the presence of σ in the magnitude condition. Once λ^* values are detected, the argument condition from Eq. (8) is used to solve for the delay σ^*.

It is clear that existence of solution pairs (σ^*, λ^*) and ultimately the stability assessment is dependent on the *geometry* of $\chi(\lambda)$. In the following, we will study this geometry to characterize the stability of the supply network.

3.3.2 Geometric Classification of the Curve $\chi(\lambda)$. Notice that when $\lambda\to\infty$, the curve χ reaches to the origin of the complex plane. The properties curve χ exhibits for finite λ are classified below.

LEMMA 1. *For any eigenvalue $0<J_{ii}<1$, there can be at most two distinct $\lambda=\lambda^*>0$ values for which $\chi(\lambda^*)=1$.*

Proof. If $\chi(\lambda^*)=1$, then from Eq. (9) we have

$$f(\lambda^*) = \lambda^{*4} + (2AJ_{ii} + (BJ_{ii}-1)^2 - B^2)\lambda^{*2} + A^2(J_{ii}^2-1) = 0 \quad (10)$$

Since the coefficients of s in the characteristic equation $F(s,\sigma)$ are real, λ^* and $-\lambda^*$ satisfy Eq. (10) and $F(s,\sigma)=0$ concurrently. For

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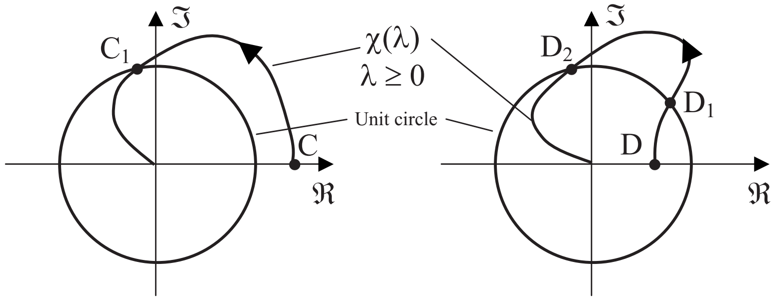

Fig. 3 Two possible scenarios for the geometric interaction between $\chi(\lambda)$ and the unit circle as per Lemma 1. Only subfigure on left is admissible as per Lemma 2.

this reason, it is sufficient to inspect Eq. (10) for the quadratic form $(\lambda^*)^2$. This equation has two $(\lambda^*)^2$ solutions, which are distinct since the discriminant of Eq. (10) is nonzero. ∇

Possible scenarios suggested by the above lemma are sketched in Fig. 3. There can be either one (point C_1) or two intersection points (points D_1 and D_2) between $\chi(\lambda)$ and the unit circle. Furthermore, points C and D represent $\chi_0=\chi(0)$ and the arrows indicate increasing λ directions from 0 to $+\infty$. Clearly, the intersection points C_1, D_1, and D_2 will yield the solution pair (λ^*,σ^*) we are seeking for.

At this point, we wish to partition the discussion with respect to the eigenvalues of $\mathbf{J}$.

Eigenvalues $0<J_{ii}<1$. The curve $\chi(\lambda)$ always initiates outside the unit circle on the real line, i.e., $\chi_0>1$, left sketch in Fig. 3. Since $\chi(\lambda)$ reaches to zero when $\lambda\rightarrow+\infty$, it is guaranteed that $\chi(\lambda)$ intersects the unit circle at least once, for instance, at point C_1.

LEMMA 2. *When eigenvalues are strictly less than 1,* $0<J_{ii}<1$, *the curve* $\chi(\lambda)$ *intersects the unit circle only at one point. In other words, there exists only one* $\lambda=\lambda^*$ *that satisfies* $\chi(\lambda^*)=1$.

Proof. Since discriminant in Eq. (10) is strictly positive, $(\lambda^*)^2$ solutions are real. Furthermore, the coefficient of λ^0 term in Eq. (10) is strictly negative; thus two $(\lambda^*)^2$ solutions of Eq. (10) are in opposite sign. Only $(\lambda^*)^2>0$ solution is admissible. ∇

According to the proof above, the only scenario that is possible is the subfigure on the left in Fig. 3. Let us denote by $\bar{\sigma}$ the minimum positive delay at which this scenario occurs, $\bar{\sigma}=\min(\sigma^*)$, where $\chi(\lambda^*)=1$ and $f_i(\lambda^* j,\sigma^*)=0$.

Eigenvalues $J_{ii}=1$. This is the case setting *maximum input rates* in the supply network. With $J_{ii}=1$, the curve $\chi(\lambda)$ initiates at point $\chi_0=1$, which is also on the unit circle. The following lemma characterizes the geometric interaction of the curve $\chi(\lambda)$ and the unit circle.

LEMMA 3. *For* $J_{ii}=1$, *the delayed supply network dynamics is at best marginally stable for any selection of A and B parameters. Moreover, if* $B-A<1/2$, *then there exists no imaginary axis crossings of the delayed supply network for some* $(\lambda^*)^2>0$ (*left subfigure in Fig. 4). If* $B-A>1/2$, *then the supply network dynamics with transportation delay exhibits one imaginary axis crossing for* $(\lambda^*)^2>0$ *(right subfigure in Fig. 4).*

Proof. Substituting $J_{ii}=1$ into Eq. (10) yields

$$f(\lambda^*) = \lambda^{*4} + (1+2A-2B)\lambda^{*2} = 0 \qquad (11)$$

where one $(\lambda^*)^2$ solution is zero and it always exists for any σ, see also Eq. (8). It indicates that characteristic equation possesses a zero pole, $s=\lambda^* j=0$, which makes the supply network dynamics *at best marginally stable* for any σ.

The second $(\lambda^*)^2$ solution from Eq. (11) exists (it is positive) only when $B-A>1/2$. ∇

Remark: Invariant root(s) at the origin. As mentioned above, when $J_{ii}=1$ the presence of a zero pole at the origin of $\mathbb{C}$ is independent of delay σ. This is an invariant root; however, its presence may initiate a second pole at the origin (hence double roots of the characteristic function at $s=0$) if the first derivative of the characteristic function with respect to s at $s=0$ becomes zero. This corresponds to

$$\left(\frac{df_i(s,\sigma)}{ds}\right)_{s=0} = 1 + B(1-J_{ii}) - A\sigma = 0$$

from which delay σ becomes $\sigma=\sigma_0=(1+B(1-J_{ii}))/A$. Clearly, delay $\sigma_0>0$ exists. When $\sigma=\sigma_0$ and $J_{ii}=1$, characteristic function possesses two poles at the origin. Since only one root at the origin is known to be *invariant*, the second root will cross to $\mathbb{C}_+$ causing instability as σ infinitesimally increases from σ_0.

3.3.3 Stability Assessment. Since $\chi(0)=1/J_{ii}\geq 1$, the curve $\chi(\lambda)$ never initiates inside the unit circle for $\lambda=0$. Consequently, $\chi(\lambda)$ is guaranteed to intersect the unit circle (since $\chi(\lambda)\rightarrow 0^+$

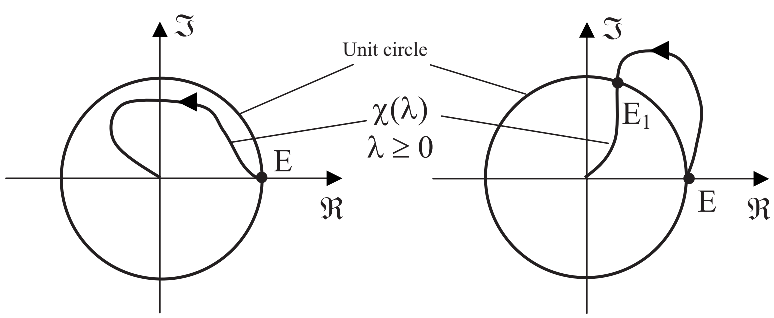

Fig. 4 Interaction of the curve $\chi(\lambda)$ and unit circle when J_{ii}=1

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when $\lambda \to \infty$) and stability to instability transition will occur. Let us summarize below the obtained results:

PROPOSITION 1. *The asymptotic/marginal stability features of the supply network are as follows*.

Case (i). Asymptotic stability is maintained with $\sigma \in [0, \bar{\sigma})$ when $0 < J_{ii} < 1$. When $\sigma = \bar{\sigma}$, a pair of characteristic roots is on the imaginary axis at $s = \mp \lambda^* j$. This is the case when the dynamics becomes a perfect oscillator at frequency λ^*.

Case (ii). Marginal stability is maintained with $\sigma \in [0, \min(\bar{\sigma}, \sigma_0))$ when $J_{ii}=1$ and $B-A>1/2$. A pair of characteristic roots $s = \mp \lambda^* j$ is on the imaginary axis at $\sigma = \bar{\sigma}$ or a pair of double roots is at the origin of $\mathbb{C}$ ($\lambda^*=0$) at $\sigma_0 = (1+B(1-J_{ii}))/A$. Smaller delay determines the stability range on σ.

Case (iii). Marginal stability is maintained with $\sigma \in [0, \sigma_0)$ when $J_{ii}=1$ and $B-A<1/2$. A characteristic root may only move to $\mathbb{C}_+$ across the origin at delay $\sigma_0 = (1+B(1-J_{ii}))/A$.

3.4 Discussions on Stability Analysis With Multiple Delays. The analysis presented above and its connection with supply network dynamics exploit frequency sweeping tools in assessing the stability analysis. Decoupling of delay σ and frequency λ serves beneficial in achieving this. When one considers multiple delays in supply network dynamics, frequency sweeping ideas can still be adapted to the context of the problem following similar lines as covered in Sec. 3.3. However, the decoupling idea may not be possible or it needs to be supplemented with different approaches, see, for instance, Refs. [23,32], in order to obtain stability maps of the dynamics. Let us summarize in this subsection possible directions one may follow in analyzing stability of the supply network in presence of multiple delays, σ_v, $v=1, \ldots, \bar{v}$, where $\bar{v}$ is the number of delays (due to the transportation lines, decision-making activities, and production lead times). Inspecting the structure of Eq. (3), the general form of the characteristic equation similar to Eq. (7) is given by

$$F(s, e^{-\sigma_v s}, A, B, c_{i\mu}, d_{i\mu}) = P_0(s, A, B, c_{i\mu}, d_{i\mu}) + P_1(s, e^{-\sigma_1 s}, \ldots, e^{-\sigma_{\bar{v}} s}, A, B, c_{i\mu}, d_{i\mu}) = 0 \quad (12)$$

where $P_1(s, e^{-\sigma_1 s}, \ldots, e^{-\sigma_{\bar{v}} s}, A, B, c_{i\mu}, d_{i\mu})$ is a polynomial in s with coefficients in terms of $e^{-\sigma_v s}, A, B, c_{i\mu}, d_{i\mu}$. Polynomial $P_0(s, A, B, c_{i\mu}, d_{i\mu})$ does not carry any delay terms and it has the largest power of s in Eq. (12), which is $2n$.

Methods that assess stability with respect to small number of delays $\bar{v} \le 3$ are not extendable to cases with large number of delays $\bar{v} > 3$ due to the increasing complexity of detecting the stability regions (as pointed out by the *NP*-hardness [21] character of the problem). Extensive research effort in the field is the evidence of this bottleneck [26–29,35,38,42–48]. Nonconservative methodologies for two delay cases are found in Refs. [32,38,47,49–52] and numerical algorithms with sufficiently small approximation tolerances are available in Refs. [36,53,54]. Except few case specific techniques [34,49,51,52], even the general treatment of three-delay problems is still open. Recently developed ACFS [22] is a *new* venue that can remove these limitations. Implementing ACFS, a *nontrivial* four delay supply network example is added to Sec. 5 to motivate future work along these lines. It is important to note that complications faced in the field due to delays also reflect to supply network literature, where we only observe treatments with a single delay, see Refs. [3,4,9] and references therein.

4 Sensitivity of Bullwhip Effects Against Pure Delays

After establishing the stability, one also needs to guarantee that *bullwhip effects* will be avoided within the supply network. For this, the nonhomogeneous part of Eq. (4) is taken into account as a harmonic oscillator, $h_i(\tau) = h_i^0 e^{j\alpha\tau}$, where α denotes the excitation frequency and h_i^0 is the constant amplitude of the excitations. We are interested how the amplitudes of the ith node in the supply network varies as a function of α and the disturbance from the $i+1$st node, $i=1, \ldots, n-1$.

This analysis is known as frequency response analysis in control theory [55]. It is performed in steady state, when the transient regime of the dynamics fades away and is negligible (thus no effects of initial conditions). Since the dynamics is linear, its response in steady state to a periodic excitation is known to be in the following periodic form, $\mu_i(\tau) = \mu_i^0 e^{j(\alpha\tau - \beta_i)}$ with a constant phase shift β_i and a constant amplitude of the output oscillations μ_i^0. This yields the amplitude μ_i^0 of the state $\mu_i(\tau)$ as a function of excitation frequency α and delay σ,

$$\mu_i^0(\alpha, \sigma) = \sqrt{\frac{(b_i \mu_{i+1}^0)^2[(B\alpha)^2 + A^2] + h_i^0 H_i + (h_i^0)^2}{p_0(\alpha) + 2p_c(\alpha)\cos(\sigma\alpha) - 2p_s(\alpha)\sin(\sigma\alpha)}} \quad (13)$$

where

$$p_0(\alpha) = \alpha^4 + \alpha^2(B^2(1 + J_{ii}^2) + 2J_{ii}(A - B) + 1) + A^2(1 + J_{ii}^2)$$

$$p_c(\alpha) = \alpha^2(B - A - B^2 J_{ii}) - A^2 J_{ii}$$

$$p_s(\alpha) = \alpha[\alpha^2 B + A]$$

$$H_i = 2b_i \mu_{i+1}^0 [A \cos(\beta_{i+1}) + B\alpha \sin(\beta_{i+1})]$$

Notice that analyzing bullwhip effects makes sense only if the dynamics is asymptotically stable for a given delay σ^*. This corresponds to case (i) of Proposition 1. The bullwhip effects are avoided for this given delay σ^*, if the following supremum condition holds:

$$\sup_{\alpha \in \mathbb{R}, \sigma^* \in [0, \bar{\sigma}), 0 < J_{ii} < 1} \left(\frac{\mu_i^0(\alpha, \sigma^*)}{\max(\mu_{i+1}^0, h_i^0)} \right) < 1$$

If, on the other hand, there exists an excitation frequency α^* for which the above inequality does not hold, then bullwhip effects will be observed. Amplitudes of oscillations from one node to another will increase at excitation frequency α^*.

Since the numerator of $\mu_i^0(\alpha, \sigma)$ is independent from delay σ, it is relatively easy to construct some analytical tools to compare the sensitivity of bullwhip effects with and without delays. For this, one can neglect the square root sign in Eq. (13) and analyze the denominator with $\sigma=0$ (denote by Δ_0) and $\sigma \neq 0$ (denote by Δ_σ). Since $\Delta_0 = \Delta_\sigma$ when $\sigma=0$, one can express Δ_σ as $\Delta_\sigma = \Delta_0 + \bar{\Delta}$, where $\bar{\Delta}$ is the change in Δ_0 (delay-free case) when delays are incorporated,

$$\bar{\Delta} = 2p_c(\alpha)(\cos(\alpha\sigma) - 1) - 2p_s(\alpha)\sin(\sigma\alpha)$$

If $\bar{\Delta} < 0$, then bullwhip effects are more prone to occur in the presence of delays. Let us briefly elaborate on $\bar{\Delta} < 0$ condition. Notice $(\cos(\alpha\sigma) - 1) = -2\sin^2(\sigma\alpha/2) \le 0$. One can rewrite inequality $\bar{\Delta} < 0$ as

$$\sin(\sigma\alpha/2)(p_c(\alpha)\sin(\sigma\alpha/2) + p_s(\alpha)\cos(\sigma\alpha/2)) > 0$$

which can be shown to be valid in some excitation frequency range(s) for given delay σ^* and supply network parameters A, B, and J_{ii}.

5 Case Study

Let us analyze the supply network dynamics in Eq. (6) with $A=0.2$, $B=0.1$, and $b_i=1$. The aim is to calculate maximum allowable transportation delay $\bar{\sigma}$ below which the supply network maintains its asymptotic stability. Since time t is scaled by T (time constant of inventory dynamics), delay σ in this section represents the transportation time scaled by T.

5.1 Stability Analysis. For the stability of the supply network with respect to transportation delay σ, we follow the stability

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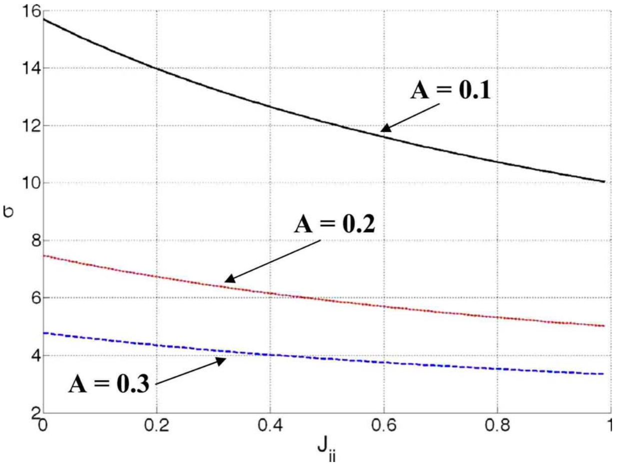

Fig. 5 Stability boundaries of the supply network for various A values, where $B=0.1$. The stability region is the area below the curves. Input rate: $0<J_{ii}<1$.

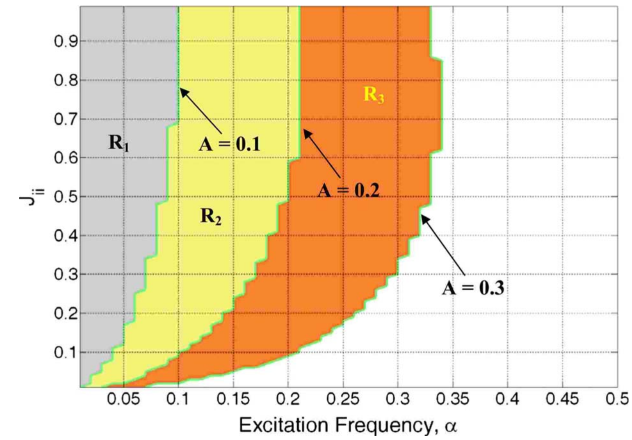

Fig. 7 Contour plot of $|\mu_i^0(\alpha,\sigma^*)|=1$ with respect to $0.01<J_{ii}<0.99$ and to A when $B=0.1$. Delay is $\sigma^*=0.5$. Bullwhip effects occur in regions R_1, $R_1\cup R_2$, and $R_1\cup R_2\cup R_3$ for $A=0.1$, $A=0.2$, and $A=0.3$, respectively.

analysis technique presented in Sec. 4. We also incorporate uncertainties. Let us assume that $A=0.2\mp 0.1$. The arising stability analysis results are depicted in Fig. 5 in the parameter space of J_{ii} and σ for different A values. In Fig. 5, the stability regions are below the curves depicted. For instance, for $J_{ii}=0.6$, the maximum allowable transportation delay $\bar{\sigma}=3.75$ when $A=0.3$. Notice also that for the choice of a fixed delay and A, the stability region narrows as J_{ii} increases. Furthermore, within the range of J_{ii}, increase in A from 0.1 to 0.2 and from 0.2 to 0.3 narrows down the stability regions.

In Fig. 6, we investigate the stability regions for the nominal value of A, $A_{\text{nom}}=0.2$, when $B=0.2\mp 0.1$. This figure is a contour plot of the transportation delay σ with respect to J_{ii} and B. The contours represent the iso-σ locations in the plane of $J_{ii}-B$. The transportation delay σ varies from 3.81 to 8.43, and the corresponding σ of each contour curve is labeled in the figure.

We conclude from Fig. 6 that larger input rates J_{ii} limit allowable transportation delays. Furthermore, in order to guarantee robustly stable supply network independent of delay effects, delay should be less than 3.81, which is 381% of the relaxation time T.

5.2 Bullwhip Effects. Once the stability of the supply network is guaranteed, bullwhip effects can be analyzed. We will adhere to the stability analysis performed in Sec. 5.1. In order to keep the presentation simple, external disturbances due to h_i^0 are taken to be zero, and quasinodes are assumed to exhibit harmonic oscillations.

In the sequel, we select $\sigma^*=0.5$ (50% of T), which guarantees the stability of the supply network, see Fig. 5. Then, for $0<J_{ii}<1$ and $B=0.1$, we analyze bullwhip effects for three different A values, $A=0.1$, $A=0.2$, and $A=0.3$. For each A and sufficiently broad range of α, we detect the amplitude of the frequency response function $|\mu_i^0(\alpha,\sigma^*)|$. Next, we obtain the contours separating J_{ii} versus α plane into two regions: one corresponding to $|\mu_i^0(\alpha,\sigma^*)|<1$ (white regions) and the other corresponding to $|\mu_i^0(\alpha,\sigma^*)|>1$ (shaded regions) in Fig. 7. When $A=0.1$, $A=0.2$,

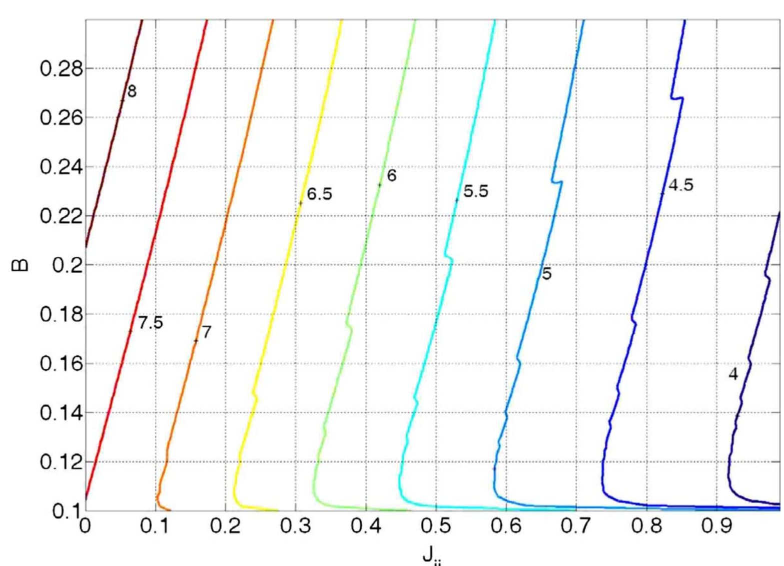

Fig. 6 Stability boundary of the supply network represented with iso-σ contours in the plane of J_{ii} versus B for the nominal value $A_{\text{nom}}=0.2$. Contour curves are labeled with their corresponding σ values. Input rate: $0<J_{ii}<1$.

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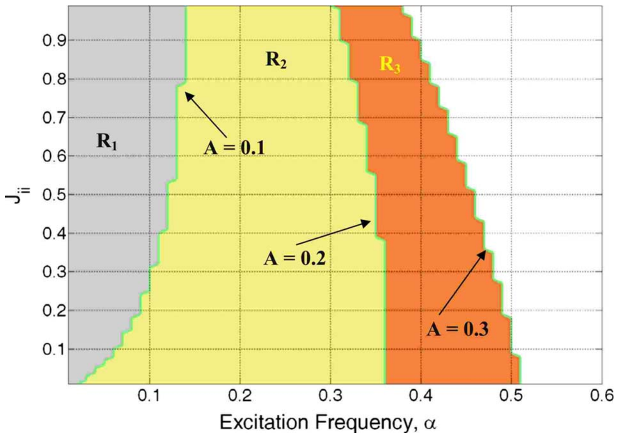

Fig. 8 Contour plot of $|\mu_i^0(\alpha,\sigma^*)|=1$ with respect to $0.01<J_{ii}<0.99$ and to A when $B=0.1$. Delay is $\sigma^*=3$. Bullwhip effects occur in regions R_1, $R_1\cup R_2$, and $R_1\cup R_2\cup R_3$ for $A=0.1$, $A=0.2$, and $A=0.3$, respectively.

and $A=0.3$, the supply network exhibits bullwhip effects in the shaded regions R_1, $R_1\cup R_2$, and $R_1\cup R_2\cup R_3$, respectively. In order to maintain a clear representation, the frequency range is shortened. For instance, we do not observe bullwhip effects for $\alpha>0.35$ and for any J_{ii} when $A=0.3$.

What is critical in this analysis is that bullwhip effects cannot be avoided for any input rate J_{ii} under these supply network parameter settings, despite the fact that delay σ^* is relatively smaller than the maximum delay that can be tolerated against losing stability (Fig. 5). In this sense, bullwhip effect becomes the dominant phenomenon, limiting the functionality of the supply network. On the other hand, the network for these settings is not prone to bullwhip effects when excitation frequency is larger, $\alpha>0.35$. These observations apparently point out the trade-off between the parameters A and delay σ and the particular way these two parameters affect both asymptotic stability and bullwhip phenomenon.

We conclude this study by offering another bullwhip analysis in Fig. 8 where the only change is in the transportation delay, $\sigma^*=3.0$. This delay still maintains asymptotic stability in the supply network; however, it is much closer to boundaries separating stability-instability regions in Fig. 5. Clearly, if one compares Figs. 7 and 8, supply network becomes more prone to exhibit bullwhip effects with transportation delay $\sigma^*=3.0$.

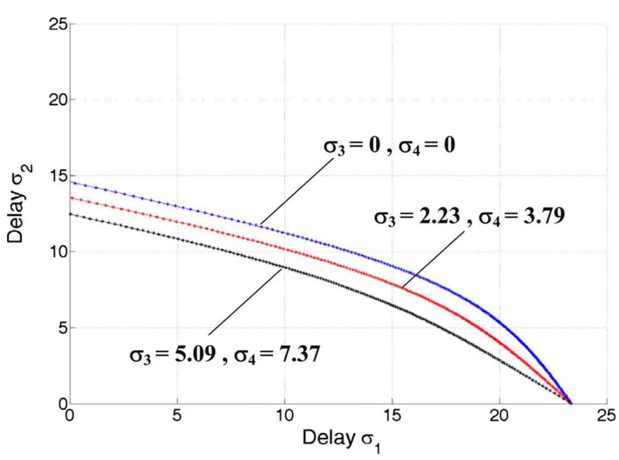

Fig. 9 Stability map of the supply network with four delays. The stable regions are entrapped by the axes and the labeled boundaries.

5.3 Supply Network With Four Delays.

Let us analyze the stability of the equilibrium of a supply network with two suppliers and two inventories. This corresponds to $i=1,2$ in Eq. (1). The features of the equilibrium dynamics is chosen as $A=B=0.1$. Notice from Eq. (1) that rows of input matrix $\mathbf{C}$ and columns of output matrix $\mathbf{D}$ sum to 1. We choose these matrices as

$$\mathbf{C}=\begin{bmatrix}0.36 & 0.64\\ 0.73 & 0.27\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}0.45 & 0.65\\ 0.55 & 0.35\end{bmatrix}$$

Four different delays, $\bar{v}=4$, are associated with each entry of matrix $\mathbf{D}$ and with these delays characteristic equation following from Eq. (5) is expressed as

$$F(s,\sigma_1,\ldots,\sigma_4)=\det\left[\begin{bmatrix}s^2 & 0\\ 0 & s^2\end{bmatrix} + s\begin{bmatrix}0.964+0.045e^{-\sigma_1 s} & 0.065e^{-\sigma_2 s}-0.064\\ 0.055e^{-\sigma_3 s}-0.073 & 0.973+0.035e^{-\sigma_4 s}\end{bmatrix} + \begin{bmatrix}0.045e^{-\sigma_1 s}-0.036 & 0.065e^{-\sigma_2 s}-0.064\\ 0.055e^{-\sigma_3 s}-0.073 & 0.035e^{-\sigma_4 s}-0.027\end{bmatrix}\right] = 0$$

which is *free* of all the assumptions that lead to Eqs. (4) and (5). The above equation after expanding the determinant becomes very complicated for analyzing stability,

$$\begin{aligned}&s^4+(0.045e^{-\sigma_1 s}+0.035e^{-\sigma_4 s}+1.937)s^3+10^{-2}[(8.8785e^{-\sigma_1 s}\\ &\quad+0.4745e^{-\sigma_2 s}+0.352e^{-\sigma_3 s}+6.874e^{-\sigma_4 s}-0.3575e^{-(\sigma_2+\sigma_3)s}\\ &\quad+0.1575e^{-(\sigma_1+\sigma_4)s}+87.03)s^2+(4.257e^{-\sigma_1 s}+0.949e^{-\sigma_2 s}\\ &\quad+0.704e^{-\sigma_3 s}+3.248e^{-\sigma_4 s}+0.315e^{-(\sigma_1+\sigma_4)s}-0.715e^{-(\sigma_2+\sigma_3)s}\\ &\quad-7.04)s-0.1215e^{-\sigma_1 s}+0.4745e^{-\sigma_2 s}+0.352e^{-\sigma_3 s}\\ &\quad-0.126e^{-\sigma_4 s}+0.1575e^{-(\sigma_1+\sigma_4)s}-0.3575e^{-(\sigma_2+\sigma_3)s}-0.37]=0\end{aligned}$$

The characteristic equation free of delays has four stable roots at $s_1=s_2=-1$ and $s_{3,4}=-0.0085\mp 0.004213j$ and we wish to reveal stability maps that display how much of delays the network can accommodate before losing stability. In order to offer a clear representation of the stability maps, we present them on the plane of σ_1 versus σ_2 while we independently choose σ_3 and σ_4. For three different numerical choices of (σ_3,σ_4) pairs, stability map of the supply network is obtained using ACFS [22] and it is given in Fig. 9. On this map, the closed region entrapped by $\sigma_1=0$ and $\sigma_2=0$ and the respective *stability boundary* labeled with (σ_3,σ_4) is the asymptotically stable operation region of the supply network on the plane of σ_1 versus σ_2. Note that each boundary in Fig. 9 can be obtained in less than 1 s on a standard Pentium 4 processor PC with 3.0 GHz CPU speed and 2 Gbyte RAM.

We finalize this case study with a simulation performed in MATLAB. From Fig. 9, $\sigma_3=2.23$ and $\sigma_4=3.79$ are taken and a stable operation point is chosen as $\sigma_1=1.13$ and $\sigma_2=3.95$. For this combination of four delays, variation in inventories and variation in production rates are simulated with respect to scaled time (Fig. 10). The *variations* exhibit asymptotic stability, as expected. Furthermore, variation in inventories initializes at negative values, which intuitively requires positive variation in production rates. This is exactly what is observed in Fig. 10 where decrease in inventories is compensated by increase in production rates.

6 Conclusion

Transportation time of products in supply networks is a source of delay, which is incorporated in a delay-free mathematical model for assessing stability and analyzing bullwhip effects with respect to this delay and parameters defining the supply network. Analytical tools are developed to achieve these analysis and to interpret various behavioral characteristics of supply networks intuitively known and empirically observed in the literature. Case

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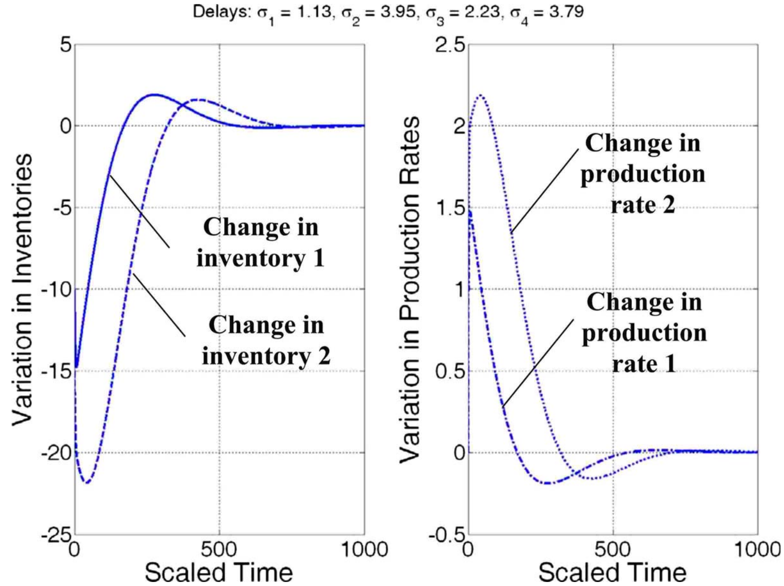

Fig. 10 Simulations of variations in inventories and production rates with respect to scaled time for a given combination of four delays. Supply network is asymptotically stable with these delays, compare with Fig. 9.

studies are provided to demonstrate the end results of the analytical approaches developed. The ultimate goal is to offer supply network managers thorough understanding of transportation delays and new tools to aid in their decision making. Future work along these lines is the cost optimization and investigation of more elaborate models with larger number of delays representing decision making, lead times, and multiple transportation lines.

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